COSMOLOGY AND HILBERT’S SIXTH PROBLEM

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SUMMARY: There have been tantalizing indications from many quarters of physical cosmology that we are living in the multiverse – a huge set of cosmological domains (“universes”). What is the structure of this larger whole is an entirely open problem on the interface between physics and metaphysics. A goal of the present paper is to draw attention to the connection between this problem and an old and celebrated puzzle in mathematical physics. Among the unresolved problems David Hilbert posed in 1900 as a challenge for the dawning century, none is more philosophically controversial than the Sixth Problem, requiring the axiomatization of physical theories. In the new century and the new millennium, this problem has remained a challenge, usually swept under the rug as “not belonging to mathematics” (as if that impacts its epistemical status) or simply “unresolved”. Recent radical ontological/cosmological hypothesis of Max Tegmark, identifying mathematical and physical structures, might shed some new light onto this allegedly antiquated subject: it might be the case that the problem has already been solved, insofar we have formalized mathematical structures! While this can be seen as “cutting the Gordian knot” rather than patiently resolving the issue, we suggest that there are several advantages to taking Tegmark’s solution seriously, notably in the domain of (future) physics of the observer.

Key words. History and philosophy of astronomy – Cosmology: theory – Astrobiology

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1. INTRODUCTION: THE SIXTH PROBLEM

The celebrated problems of David Hilbert, posed in the original form in 1900-1902 as the major problems for the incoming 20th century, have ramifications far beyond the history of mathematics and philosophy of science (for an accessible review, see Gray 2000). While some of the total of 24 problems have been re-

…a librarian of genius to discover the fundamental law of the Library. This thinker observed that all the books, no matter how diverse they might be, are made up of the same elements: the space, the period, the comma, the twenty-two letters of the alphabet. He also alleged a fact which travelers have confirmed: In the vast Library there are no two identical books. From these two incontrovertible premises he deduced that the Library is total and that its shelves register all the possible combinations of the twenty-odd orthographical symbols (a number which, though extremely vast, is not infinite). Everything: the minutely detailed history of the future, the archangels’ autobiographies, the faithful catalogues of the Library, thousands and thousands of false catalogues, the demonstration of the fallacy of those catalogues, the demonstration of the fallacy of the true catalogue, the Gnostic gospel of Basilides, the commentary on that gospel, the commentary on the commentary on that gospel, the true story of your death, the translation of every book in all languages, the interpolations of every book in all books.

Jorge Luis Borges (1999)
solved in the course of the major steps forward, even revolutions in mathematics (e.g., #10 on the solvability of polynomial Diophantine equations, or #19 on the analytic nature of solutions of regular problems in the calculus of variations, both with enormous practical applications in many fields), many are still unresolved, almost 120 years after Hilbert’s challenge. Perhaps none is more confusing than the #6, in Hilbert’s German original, Mathematische Behandlung der Axiome der Physik, and usually known in compact form as “axiomatization of physics” or “axiomatization of the physical theories”. The original elaboration reads (Hilbert 1902):

To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics. . . As to the axioms of the theory of probabilities, it seems to me desirable that their logical investigation should be accompanied by a rigorous and satisfactory development of the method of mean values in mathematical physics, and in particular in the kinetic theory. Bördtzen’s work on the principle of least action suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomicistic view to the laws of motion of continua.

As with other Problems, this setup tells much about Hilbert’s time, dominant worldview, and Hilbert’s philosophical views (Sauer 1999, Renn and Stachel 2007, Slemrod 2013). The most comprehensive review of the Sixth Problem has so far been provided by the distinguished historian of science Leo Corry (1997, 2004). One point of importance for our discussion here, namely that, as Corry notes: “[t]his problem differs in an essential way from most others in the list... it is more of a general task, than a specific mathematical problem.” (Corry 1997, p. 84)

Therefore, we are somewhat entitled to expect a different kind of solution as well. We should not allow our historical knowledge, especially that of Hilbert’s commitment to formalism and finitism, to confuse us regarding legitimacy and nature of possible solutions to the Sixth. In Table 1, we see the existing views on the status of the Sixth Problem as options (i), (ii), (iii), and we tentatively add the option (iv) of the special solution discussed in the rest of this paper.

There is a bunch of historical results usually cited in connection with the Sixth Problem. In particular, the work of Hilbert himself, as well as his best assistant in Göttingen, Emmy Noether, on differential geometry and fundamental symmetries, has been repeatedly advertised as setting the groundwork for modern mathematical physics. Cartan’s formalization of Newtonian theory of gravity in 1924, attempting to formalize general relativity in terms of the most general covariance principles, as well as Kolmogorov’s axiomatization of the probability theory in 1933 are also examples of advances, conventionally understood, in the general direction indicated by the Sixth Problem (for example, see Norton 1993). Even more pertinent is the project of Wightman and Haag, inspired by John A. Wheeler, of axiomatization of the quantum field theory, undertaken in 1960s and 1970s (Wightman 1976). Some researchers maintain that the modern pretenders to the “Theory of Everything” such as the string/M-theory which are highly abstract and mathematical in the first place could be put on completely formal, axiomatic basis, although clear are still very remote (e.g., Dawid 2013); and there are many opponents of the entire project (e.g., Smolin 2006, Ellis and Silk 2014).

Even a cursory analysis of these historical results indicates that these fail far short of the clear solution to the Sixth Problem. Fields such as probability theory or the calculus of variations are nowadays, in contrast to Hilbert’s time, firmly regarded as parts of mathematics, so their axiomatization is neither surprising nor could be regarded as the answer to Hilbert’s query. Even if formalization of contemporary theories such as M-theory proceeds successfully (prospects for which are unclear at present), and provided that it really describes the deepest and the most general level of physical reality, this still does not preclude the search for other possible solutions. After all, solutions to many other problems, including some of Hilbert’s, proceed in two possible directions: either proving a general conjecture, or finding a counterexample (Matyjaśewieć proved a general conjecture resolving the 10th Problem; Nagata constructed a counterexample to resolve the 14th Problem). The formulation of the Sixth implies that such axiomatization is possible, which is a point of contention itself for many physicists.

For instance, Richard P. Feynman would be one of the skeptics on this issue. His well-known quote about the “next great era of awakening of human intellect” (emphasis M. M. Č.) in physics as understanding the qualitative content of the equations (Feynman et al. 1964, vol. II, p. 41-12) is reasonably interpreted as implying that we do not have insight into such content as of now and with the current methods and approaches. See also his skeptical remarks about the contemporary mathematical physics in Feynman (1965), esp. Chapter 2. Not much has happened in the last 50+ years which would likely sway his opinion.

In the rest of this note, I shall try to defend a radical view that viable solution to the Sixth Problem is provided by a recent controversial hypothesis of the Swedish-American cosmologist Max Tegmark. This solution need not be regarded as unique, has not been anticipated, arguably is not in accordance with either Hilbert’s or any other historical view on the problem, and might look as a cop-out to some – but there are many inherent benefits in opening the discussion about it, starting with showing how philosophically important questions persist in contemporary theoretical physics and cosmology.

2. MATHEMATICAL UNIVERSE HYPOTHESIS

In several papers and a book in the period 1998–today, Tegmark advances a radical idea that there is no ontological difference between physical and mathematical structures.1 Scientists have for quite a long time struggled with Wigner’s “unreasonable effective-
ness of mathematics in the natural sciences” (Wigner 1960), as well as the related issue why there are some mathematical structures seemingly privileged to explain physical structures (e.g., Hilbert’s space of quantum states), while others have no such “real world” applications (e.g., Banach spaces). Tegmark simply cuts the Gordian knot by asserting that all mathematical structures are realized somewhere, so there is no distinction between physical and mathematical reality. Of course, the fact that we perceive only some physically realized mathematical structures testifies upon the strength of observation-selection effects: as “self-aware substructures”, we observe only those structures supporting sufficient complexity, stability, and other physical pre-requisites for observership. In other parts of this radical Platonist multiverse, other mathematical structures are realized without observers around to notice that. In a sense, Tegmark’s hypothesis – often going under the title of the Mathematical Universe Hypothesis (MUH) – offers the ultimate logical conclusion to the Galilean project of reading nature as the book written in the language of mathematics.

It is an understatement that Tegmark’s hypothesis has been and remains controversial. Although Tegmark has argued that MUH has observable consequences, most critics have disagreed.2 Without going into extremely complex issues of possible computable measures in such a large multiverse (sic!), it is important to state that the Level IV mathematical multiverse is arguably the most extreme position within the corpus of “postmodern cosmology” (cf. different positions within the same volume in Carr 2007, Kragh and Longair 2019). The origin of the label is located with Tegmark’s popular review in Scientific American (Tegmark 2005), where he offered a tentative classification scheme for easier thinking about the grand cosmological ensembles. The Level I would be an extension of our universe beyond the cosmological horizons, something which is uncontroversial even in the standard Friedmann models. The Level II multiverse would encompass all cosmological domains (“universes”) originating through cosmological inflation, or a similar symmetry-breaking process, in the early universe. On a deeper level still, if many-worlds interpretations/theories of quantum mechanics – most notably, Everett’s “no collapse” theory – are valid, the totality of all branches of the wavefunction of the universe would constitute the Level III multiverse. Finally, any structure larger than that would belong on the Level IV – and in particular this applies to Tegmark’s own “ultimate ensemble” implied by MUH. (Note that each level contains lower levels as special cases in which at least one global symmetry is broken.) This hierarchical scheme helps to explain why MUH is met with agnosticism, or even incredulity, by many proponents of other multiverse schemes – but it has undoubtedly “stirred the pot” and brought further exposure and visibility to cosmological and metaphysical issues.3

From the point of view of this paper, MUH offers a simple, obvious, and yet far-reaching answer to the Sixth Problem:

There is no independent formalization of physics – and it cannot exist. The reason is that there is no independent physical world. Axioms of mathematics are the true “laws of physics” as well; consequently, the axiomatization of physics has already been achieved insofar and in the same manner as the mathematical realm is axiomatized.

There are two steps here: (i) realization that the apparent physical world reflects a deeper mathematical structure, and (ii) using this insight to address the Sixth Problem. Obviously, a separate physical description is entirely redundant, since all information is contained in the mathematical description; this is obvious in systems with very high degree of symmetry (e.g., crystals), but is indeed present in all systems in the universe. In other words, just as we can derive particular secondary quality of a piece of quartz, say its color, from its underlying symmetry group, thus could all properties of all objects be derived – according to Tegmark – from the underlying mathematical structures. Of course, we lack the required level of sophistication at present, but there is no substantial obstacle in the world itself.

(Of course, one can still hold different opinions on the role and meaning of axioms in the mathematical realm, just as physicists and philosophers of physics often have differing views and opinions on the role of

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3For some of the novel work motivated by Tegmark’s hypothesis, see Yampolskiy (2017), Hamlin (2017).
laws of physics in the physical world; see, for instance, Maudlin 2007.)

Perhaps the most predictable objection to the present suggestion is that this certainly is not what Hilbert meant when he posed the problem. Putting aside a pertinent question whether we should care at all about the “original intent” in science, it is hardly the case that other Hilbert’s problems have been solved entirely to his liking. We might only speculate what would he thought about the validity of computer-assisted proofs (18th Problem) – and it is a matter of historical record that he was personally dissatisfied with the steps taken by Gödel and Gentzen in resolving the Continuum Hypothesis (1st Problem), and in the issue of proving consistency of arithmetic (2nd Problem). This is hardly surprising. With the passage of time, it is only natural that the original expectations will be more and more removed from both contemporary reading of the problem situation and the reality of our scientific insights which are much broader in their scope.

As far as the problem of the consistency of arithmetic in Gödel showed in 1931 that no proof of its consistency can be carried out within arithmetic itself (e.g., Dawson 1997). Nobody doubts that this result has been an important step in resolving the problem: the controversy exists only to what extent the former are compatible with the latter. In particular the solution of the Second Problem as offered by Gödel’s second incompleteness theorem, contains an analogy with the present proposal: Hilbert asked for proving of the consistency of arithmetic and Gödel showed in 1931 that no proof of its consistency can be carried out within arithmetic itself (e.g., Dawson 1997). Nobody doubts that this result has been an important step in resolving the problem: the controversy exists only to what extent the former is necessary for the solution (e.g., selecting a richer formal system within which the consistency could be proven).

In addition, Tegmark’s solution neatly avoids an important problem plaguing all partial axiomatisations, namely the (lack of) consistency of axioms in different local formal systems. As Corry (1997, p. 122) points out:

Modeling this research on what had already been done for geometry meant that not only theories considered to be closer to “describing reality” should be investigated, but also other, logically possible ones. The mathematician undertaking the axiomatization of physical theories should obtain a complete survey of all the results derivable from the accepted premises. Moreover, echoing the concern already found in Hertz and in Hilbert’s letters to Frege, a main task of the axiomatization would be to avoid that recurrent situation in physical research, in which new axioms are added to existing theories without properly checking to what extent the former are compatible with the latter.

Insofar as Gödelian uncertainty persists in the foundations of mathematics, we may still worry about consistency even under MUH. However, we get rid of the additional worry related to possible incoherence of specifically chosen axioms for physical theories.

4Consider examples such as Tycho’s being ardent on proving that Earth does not move, or Einstein’s insistence in 1917 that the universe on large scales must be static which led to his introduction of the cosmological constant.

3. THE SIXTH PROBLEM IN RELATION TO OBSERVERSHIP

When he posed the Problems, Hilbert could not have been aware of the debate which will become central to most of philosophy of science in the course of the 20th century: the role of observer in physical reality. In contexts of both quantum physics and modern cosmology, observership has become an object of scrutiny: both the orthodox Copenhagen interpretation of the wave function and anthropic reasoning in cosmology ascribe important role to particular qualities of emergent or evolved observers. Note that Hilbert’s notion of physics in the formulation of the Sixth Problem tacitly implies the observer: kinetic theory of gases, hydrodynamics and other examples he mentions are such as they are because we observe them in a habitable universe. Those low-level effective laws valid near “our” vacuum state are selected from the underlying true laws valid at all energies by the filter that they support observers like us. In addition, those processes which occur on, for example, drastically different spatial scales than the one of our immediate sensory experiences (e.g., interactions of molecules in the kinetic theory of gases) are clearly contingent on the evolutionary fact that entities qualifying as observers must have minimal spatial size in comparison to the entities in question.

Under MUH, our cosmological domain (“the universe”) is embedded first into the subset of all habitable universes (“the Archipelago of Habitability”), and then into the total Level IV multiverse, described by all mathematical structures. The question which naturally arises is what true subset of structures is necessary and sufficient for description of both the Archipelago and the universe? What is the true “mathematics of the observer”? We clearly lack answers to these questions: both Tegmark (1998) and authors on the anthropic selection of habitable universes within the multiverse (e.g., Susskind 2006, Gleiser 2010, Soler Gil and Alfonseca 2013) mention some of the desiderata (supporting threshold complexity, stability, predictability, etc.), but we are still far from the detailed, quantitative theory defining all necessary and sufficient conditions. The arrow of time, for example, as far as necessary for the existence of life and observers, could be obtained in a refashioned Boltzmann-Schuetz anthropic selection picture (Price 1996, Ćirković 2003).

While Tegmark’s “cutting of the Gordian knot” resolves the problem of empirical fine-tunings of our universe, it can be argued that MUH is seriously incomplete without more precise, quantitative account of the prerequisites for observership. Among other things, this points to the ambiguity inherent in the proposed resolution of the Sixth Problem: while the total structure of the Level IV multiverse could have a compact description as given by the totality of mathematical structures, we are still facing uncertainty as to the substructure supporting observers, which is the relevant physics (or, even more accurately, physicses meaning different low-energy effective laws supporting observers). It might not be the sense of “relevant” in the originalist meaning of Hilbert’s views,
which would anyway be an anachronism, but something closest to it when we account for the progress made in fundamental physics and cosmology to this day.

Still more precisely, we wish to understand better how to use the “master equation” giving the probability \( p(X) \) that some observer anywhere in the multiverse measures a feature \( X \) (e.g., Carroll 2006):

\[
p(X) = \frac{\sum_n \sigma_n(X) V_n \rho_{n_{\text{obs}}}}{\sum_n V_n \rho_{n_{\text{obs}}}},
\]

where the index \( n \) labels all possible vacuum states (“physicses” or different universes in the multiverse). In current versions of string/M-theory there is a finite number of such states, although it is huge (\(10^{500}\) or so), but in principle it could be infinite. The latter case poses some interesting problems in the theory of probability, but in general, it will not preclude the usage of the master equation with appropriate weightings. \( V_n \) is the spacetime volume belonging to the universe \( n \), \( \rho_{n_{\text{obs}}} \) is the density of observers in the same universe, and the indicator function is:

\[
\sigma_n = \begin{cases} 
1, & \text{if universe } n \text{ has property } X \\
0, & \text{otherwise}
\end{cases}
\]  

In principle, \( V_n \) is calculable from our understanding of cosmological physics, although in weird enough universes it might be quite difficult to calculate in practice. It is also likely to be infinite in some or most of the universes, so an appropriate weighting procedure is certainly necessary. But, of course, the biggest uncertainty comes from the quantity \( \rho_{n_{\text{obs}}} \), the density of observers. It is usually assumed to be proportional to the density of galaxies, but that is a sort of gimmick, since obviously galaxies have vastly variable habitability even within our single universe (e.g., Dayal et al. 2015, Vukotić et al. 2016, Stojković et al. 2019). One of the foremost tasks for the theoretical astrophysics of near future is to build quantitative models which go beyond this gimmick.

In any case, the project initiated by MUH cannot be completed even in the most abstract, conceptual sense without providing means for answering tough questions on the physics of observability. These questions point to a new confluence between fundamental physics and cosmology on one side and astrophysics, evolutionary theory, and cognitive science on the other. In this sense, the Sixth Problem might become even more interdisciplinary in the foreseeable future.

4. DISCUSSION

To summarize, I have argued that Tegmark’s mathematical universe theory actually provides a solution to Hilbert’s Sixth problem. It might be implicit, it might not be the solution Hilbert and others had in mind, it might not be a likeable solution, it might not please the community, but it is a solution nonetheless. In the current rather fractured state of fundamental research it is perhaps the only solution of the Sixth on the horizon.

It certainly is not exclusive in the sense that it precludes any further work on the subject matter. On the contrary, if anything, Tegmark’s solution provokes quite an interesting problem on the interface between mathematics, astrophysics, cognitive science, and philosophy, namely what is the most general class of mathematical structures supporting evolution of intelligent observers? As discussed above, there are many auxiliary questions necessary to address and clarify before the answer to this general query could be given, and these auxiliary questions are likely to engage attention of many different research profiles for quite some time to come (cf. Ćirković 2012, Ćirković and Dimitrijević 2018).

When “a librarian of genius” in Borges’s Library of Babel discovers the fundamental law of the Library, such an achievement has probably been accompanied by various kinds of conservative resistance, perhaps including the allegations that he did not understand the problem. This Newton or Einstein of the fictional world should not have been bothered by complaints that the discovery is “metaphysical”, “abstract”, “ugly”, not to mention “impractical”. Arguably, the discovery of the fundamental law need not change anything in the daily routine of most inhabitants of Borges’s universe. Local political and administrative structures could elect to ignore it entirely. All that does not, however, subtract from the importance of the discovery – if anything, it adds to the magnitude of the required intuitive leap and the elegance of the solution. Few Borgesian lessons could not be applied to the real world, however. While the architecture of our multiverse is likely to be the topic of much work in the centuries to come, a possibility that the quest will result in unexpected side benefits and resolutions should not be discounted.


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Присутни су интригантни наговештаји из многих области космологије да живимо у мултиверзуму - огромном скупу космосских домена ("универзума"). Каква је структура ове веће целине у потпуности је отворен проблем на прелазу из физике у метафизику. Циљ овог рада је да усмери пажњу на везу између овог проблема и старе и прослављене загонетке математичке физике. Међу нерешеним проблемима које је Дејвид Хилберт поставио 1900. године као изазов за наступајући век, ниједан није више филозофски контроверзан од Шестог проблема, који захтева аксиоматизовање физичких теорија. У новом веку и новом милијуму овај проблем остао је изазов који се обично гура под тешких као проблем који "не спада у математику" (као да то нарушава његов епистемолошки статус) или је просто "нерешив". Скорашње радикалне онтолошке/космолошке хипотезе Макса Тегмарка, које идентификују математичке и физичке структуре, можда могу баци ново светло на ову највише антипатриархалну тему: можда је проблем већ решен, у смислу да смо формализовали математичке структуре! Иако ово може изгледати као "пресецање Гордијевог чвора" уместо стрпљивог решавања проблема, предпочивамо да постоји више предности у узимању Тегмарковог решења озбиљно, пренетиво у домену (будуће) физике посматрача.