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HOMOGENEOUS STELLAR MODEL HAVING CHEMICAL COMPOSITION: X = 0.628 AND Z = 0.047

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SUMMARY: Presented is a model with the mass equal to one solar mass, the abundance of the hydrogen $X = 0.628$, the abundance of the helium $Y = 0.325$ and the abundance of the metals $Z = 0.047$. This model corresponds to the young stars belonging to the Population I.

1. PRESENTATION OF THE PROBLEM

We consider a model of a star having a radiative nucleus and a convective cover. Solving the problem integration must be performed both from centre and from surface and the solutions thus obtained have to be connected, so that the continuity of the considered parameters should be ensured. To give a model of the interior of a star means to determine the variations of pressure, temperature, mass and luminosity along the ray. The following equations of hydrostatic equilibrium, mass distributions, luminosity and temperature are valid for the radiative nucleus(see, e.g., Menzel and others, 1963; Aller and McLaughlin, 1965; Cox and Giuli, 1968):

$$\begin{aligned} &dP(r)/dr = (-GM(r)/r2)\rho(r), \\ &dM(r)/dr = 4\pi r^2\rho(r), \\ &dL(r)/dr = 4\pi r^2\rho(r)\varepsilon(r) \\ &dT(r)/dr = (-3/4ac)\cdot(\kappa(r)\rho(r)/T^3(r))\cdot \\ &\qquad (L(r)/4\pi r^2) \end{aligned} \quad (1)$$

$\rho(r)$ is the density at the distance r from the centre, $\varepsilon(\rho(r), T(r), X, Y)$ is the energy generation per gram per second, $\kappa(\rho(r), T(r), X, Y)$ is the opacity corresponding for the mass unity, and X, Y are the fractions of hydrogen and helium. The system (1) has the following boundary conditions in the centre of the star:

$$\begin{aligned} &M(0) = 0, \\ &L(0) = 0, \\ &P(0) = P_c =?, \\ &T(0) = T_c =? \text{ at } r = 0 \end{aligned} \quad (2)$$

The law of the gas $P(r) = (1/\mu)(k/H)\rho(r)$ $T(r)$ is valid for the whole interior.

The hydrostatic equilibrium equation as well as the mass distribution and the adiabatic equations (Menzel *et al.*, 1963):

$$\begin{aligned} &dP(r)/dr = (-GM(r)/r^2)\rho(r), \\ &dM(r)/dr = 4\pi r^2\rho(r), \\ &P(r) = K\rho(r)^{5/3} \text{ or } P(r) = K_1 T^{2.5}(r) \end{aligned} \quad (3)$$

are valid for the whole convective zone.

The system (3) has the following boundary conditions at the star surface:

$$M = M_0,\quad L = L_0,\quad T = 0,\quad P = 0 \text{ at } r = R_0 \tag{4}$$

Schwarzschild's transformations are applied to the systems (1) and (3) (Schwarzschild, 1958):

$$\begin{aligned} P(r) &= pGM^2/(4\pi R^4),\\ T(r) &= t(\mu H/k)(GM/R),\\ M(r) &= qM,\\ L(r) &= fL \text{ and } r = R \cdot x \end{aligned} \tag{5}$$

where henceforth p, t, q, x, f are dimensionless variables. To produce the energy we consider the following formula:

$$\varepsilon = \varepsilon_0 \rho(r) T^{4.5}(r) \text{ where } \varepsilon_0 = 2.8 \cdot 10^{-33} X^2 \tag{6}$$

and for opacity:

$$\kappa = \kappa_0 \rho^{0.75}(r) T^{-3.5}(r) \text{ where } \kappa_0 = 6.52 \cdot 10^{24}(Z + (X+Y)/59.3)(1+X)^{0.75} \tag{7}$$

Using Schwarzschild's transformations and the laws (6) and (7) in the systems (1) and (3), they become:

$$\begin{aligned} dp/dx &= -pq/(tx^2),\\ dq/dx &= px^2/t,\\ df/dx &= Dp^2x^2t^{2.5},\\ dt/dx &= -C(p^{1.75}f)/(x^2t^{8.25}) \end{aligned} \tag{8}$$

respectively

$$\begin{aligned} dp/dx &= -pq/(tx^2),\\ dq/dx &= px^2/t \text{ and } p = Et^{2.5} \end{aligned} \tag{9}$$

where

$$\begin{aligned} E &= 4\pi K_1 (H/k)^{2.5} G^{1.5} M^{0.5} R^{1.5} \mu^{2.5}\\ C &= (3\kappa_0/(4ac))(1/(4\pi)^{2.75})\,(k/(HG))^{7.5}\\ &\quad (LR^{1.25}/(M^{5.75}\mu^{7.5}))\\ D &= (\varepsilon_0/4)(GH/k)^{4.5}(M^{6.5}/(LR^{7.5}))\mu^{4.5} \end{aligned} \tag{10}$$

The boundary conditions become as follows:

$$\begin{aligned} &\text{at the centre: } x = 0,\ f = 0,\ q = 0,\ t = ?,\ p = ?\\ &\text{and at the surface: } x = 1,\ f = 1,\ q = 1,\ t = 0,\\ &\quad p = 0. \end{aligned} \tag{11}$$

If we start the integration of the system (8), we obtain two infinite assemblies of solutions for the nucleus, due to the possibility of choosing the values of pressure and temperature in the center. We perform another variable transformation, which will remove an infinite assembly of solutions for the radiative nucleus.

We consider:

$$\begin{aligned} &x = x_0 x^* \text{ and } t = t_0 t^*,\\ &f = f_0 f^*,\\ &p = p_0 p^* \text{ and } q = q_0 q^* \end{aligned} \tag{12}$$

where x_0, t_0, f_0, p_0, q_0 are indefinite constants. We impose the following form to the system (8):

$$\begin{aligned} dp^*/dx^* &= -p^*q^*/(t^*{x^*}^2),\\ dq^*/dx^* &= p^*x^{*2}/t^*,\\ df^*/dx^* &= p^{*2}x^{*2}t^{*2.5},\\ dt^*/dx^* &= -p^{*1.75}f^*/(t^{*8.25}x^{*2}) \end{aligned} \tag{13}$$

and thus x_0, t_0, p_0, f_0, q_0, C, D verify the system (14):

$$\begin{aligned} &q_0/(t_0x_0) = 1\\ &p_0x_0^3/(t_0q_0) = 1,\\ &C\ p_0^{1.75}f_0/(t_0^{9.25}x_0) = 1,\\ &D\ p_0^2t_0^{2.5}x_0^3/f_0 = 1. \end{aligned} \tag{14}$$

If we consider an already known chemical composition, we may calculate the value of C and D, but besides them the system (14) contains five unknown quantities, so one of them may be chosen. We have chosen $t_0 = t_c$, so $t_c^* = 1$.

Both the system (9) and the system (13) present singularities in the points where the boundary conditions are given. A difficult problem, the one of connecting the solutions should be elucidated. We have to ensure the continuity of the parameters $P(r)$, $T(r)$, $M(r)$ and $L(r)$. Three new parameters are introduced by the relations:

$$\begin{aligned} U &= d\,\log M(r)/(d\,\log r),\\ V &= -d\,\log P(r)/(d\,\log r),\\ (n+1) &= d\,\log P(r)/(d\,\log T(r)). \end{aligned} \tag{15}$$

We perform the calculations in (15) and obtain:

$$\begin{aligned} U &= 4\pi r^3 \rho(r)/M(r) = px^3/(qt) = p^*{x^*}^3/(q^*t^*),\\ V &= (\rho(r)/P(r)) \cdot (GM(r)/r) = q/(tx) =\\ &\quad q^*/(t^*x^*), \end{aligned} \tag{16}$$

and $(n+1)$ form corresponding to the radiative nucleus will become:

$$\begin{aligned} (n+1)_{rad} &= (16\pi ac/3)(GM(r)T(r)^4/(P(r)\kappa(r)L(r)))\\ &= (1/C)(qt^{8.25}/fp^{1,75}) = q^*t^{*8.25}/(f^*p^{*1.75}) \end{aligned} \tag{17}$$

We obtain $(n+1)$ form corresponding to the convective zone and we get:

$$(n+1)_{conv} = 2.5 \tag{18}$$

Pressure and temperature being continuous functions, $(n+1)$ should be a continuous function too. The convective zone begins in the point x^* where $(n+1) = 2.5$. Starting with a certain value for p_c^*, within the plane (U, V) we obtain a corresponding curve having a final corresponding value (U_i, V_i) where the radiative zone ceases to exist. Starting with a certain E we can integrate system (9) and plot the corresponding curve in the plane (U, V). But the continuity of the functions corresponding to mass and pressure requires a continuous curve in the plane (U, V). Thus, if we choose a certain E, then we may choose a value for p_c^* so that the continuity within the plan (U, V) should be obtained, but we may consider the problem the other way round as well, that is to start by choosing p_c^* and then to interpolate as against E.

We suppose that a connection of a certain E and a p_c^* has been achieved. Then we determine the constants x_0, p_0, f_0, q_0, t_0, C and D. The assumption that a connection has been achieved gives us the value of the parameters q, p, f, t at the overlaping both from surface and from centre, thus we know: x_{is}, q_{is}, t_{is}, p_{is} and x_{ic}, p_{ic}, t_{ic}, f_{ic}, q_{ic} and, as there is no energy produced within the convective zone; it follows $f_{is} = 1$, where "is" indicates that there is a value at the overlaping considered from surface, and "ic" indicates that there is a value of a parameter, considered from centre. Using (12), we have:

$$\begin{aligned} &x_{is} = x_0 x_{ic}^* \text{ and } p_{is} = p_0 p_{ic}^*,\ f_{is} = 1 = f_0 f_{ic}^*, \\ &q_{is} = q_0 q_{ic}^* \text{ and } t_{is} = t_0 t_{ic}^* \end{aligned} \tag{19}$$

which give us the values x_0, f_0, t_0, q_0. The system (14) gives us the values of C and D. We suppose that the values of C and D are calculated for a certain E and p_c^* for which a connection of the solutions has been achieved. Using the formulae of C and D given by (10), where M, R, L which stand for mass, ray, luminosity corresponding to the Sun at the present time, are considered as known data and testing with different chemical compositions we try to obtain values for C and D, equal to those resulting from the calculation. Thus, once the calculus achieved, that is a chemical composition which has been determined, it should be reconsidered until there is obtained a chemical composition as close as possible to the one determined in spectroscopy.

The formulae (1) - (19) are given in Menzel (1963).

2. THE PROBLEM SOLVED NUMERICALLY

System (13) has the following limit conditions:

$$x^* = 0,\ f^* = 0,\ q^* = 0,\ t^* = 1 \text{ and } p^* \text{ chosen} \tag{20}$$

This system has a singularity in $x^* = 0$, but system (13) admits solutions in an analytic form for each and every neighbourhood of this singularity point. These analytic solutions are prolonged by continuity in the point $x^* = 0$ as well. We note $p_c^* = p_0$, considering the $\sum a_n x^n$ solutions and imposing the condition that these series should verify (13), we obtain:

$$\begin{aligned} p(x) =& p_0 - (1/6)p_0^2 x^2 + (1/45)(p_0^3 - p_0^{5.75})x^4 + \\ & 0x^5 + A_6 x^6 + ... \\ q(x) =& (1/3)p_0 x^3 + (1/30)(p_0^{4.75} - p_0^2)x^5 + \\ & 0x^6 + B_7 x^7 + ... \\ f(x) =& (1/3)p_0^2 x^3 - ((1/15)p_0^3 + (1/12)p_0^{5.75})x^5 + \\ & 0x^6 + C_7 x^7 + ... \\ t(x) =& 1 - (1/6)p_0^{3.75} x^2 + ((59/1440)p_0^{4.75} - \\ & (3/32)p_0^{7.5})x^4 + 0x^5 + D_6 x^6 + ... \end{aligned} \tag{21}$$

The series (21) will help us in calculating the values of the solutions in four points contiguous to the origin and to the integration pass $h = 0.01$. In order to obtain the value of the solutions in the following points, we use Adams-Bashforth's extrapolation formula of the forth order:

$$\begin{aligned} V_{k+1} =& V_k + h[(55/24)f_k - (59/24)f_{k-1} + \\ & (37/24)f_{k-2} - (9/24)f_{k-3}] \end{aligned} \tag{22}$$

which allows us to calculate the solution in a certain point, if we know the values in four previous points.

Adams-Moulton's interpolation formula:

$$V_{k+1} = V_k + h[b_{-1} f_{k+1} + ... + b_3 f_{k-3}] \tag{23}$$

contains the solution V_{k+1} within the right term in the item f_{k+1}. From (22) we obtain a ${V_{k+1}}^{(0)}$ which substituted in (23) gives the possibility of obtaining ${V_{k+1}}^{(1)}$. We apply the successive approximations method and we obtain:

$$\begin{aligned} {V_{k+1}}^{(n+1)} =& V_k + (251/720)hf(t_{k+1}, {V_{k+1}}^{(n)}) + \\ & h[(646/720)f_k - (264/720)f_{k-1} + \\ & (106/720)f_{k-2} - (19/720)f_{k-3}] \end{aligned} \tag{24}$$

The process of approximation continues until $\left|{V_{k+1}}^{(n+1)} - {V_{k+1}}^{(n)}\right| < 10^{-11}$. The system (9) will be integrated under the following condition: for $x = 1$, $p = t = 0$, $q = 1$, E chosen. We perform the variable $y = 1 - x$, we denote the variable by x as well, and thus the system (9) becomes:

$$\begin{aligned} &dp/dx = pq/(t(1-x)^2), \\ &dq/dx = -p(1-x)^2/t, \\ &dt/dx = (1/2.5E)\, pq/(t^{2.5}(1-x)^2) \end{aligned} \tag{25}$$

It has a singularity in the point $x = 0$ because of t.

We still use the $\sum A_n(1-x)^n$ series method, and we obtain out of (25):

$$p(x) = (E/(2.5)^{2.5})(1-x)^{2.5} + ...$$
$$q(x) = 1 - (E/(2.5)^{2.5})(1-x)^{2.5} + ...$$
$$t(x) = (1/2.5)(1-x) + (14E/(4+25E))(1-x)^2 + ... \quad (26)$$

We use (26) in calculating the value of the solutions in one single point contiguous to 1. In order to calculate the values of the solutions in the following three points, we use Runge-Kutta's method for non-autonomous systems:

$$V_{k+1} = V_k + h[(1/6)l_1 + (1/3)l_2 + (1/3)l_3 + (1/6)l_4)$$
$$l_1 = f(t_k, V_k) \text{ and} l_2 = f(t_k + (h/2),\ V_k + (h/2)l_1)$$
$$l_3 = f(t_k + (h/2),\ V_k + (h/2)l_2)$$
$$l_4 = f(t_k + h, V_k + hl_3),\ h = t_{k+1} - t_k = 10^{-3} \quad (27)$$

Thus, we obtain the values of the solutions in four points, which allows us to continue with the predictor-corrector method. The formulae (22)-(24) appear in Moszynski (1973).

3. RESULTS AND CONCLUSIONS

As we have already stated in the first chapter, we choose a p_c^* and perform the interpolation considering different values of E until we obtain a connection within the plane (U, V), and with the help of the values C and D resulting from the calculus, we determine the chemical composition. The whole calculus is repeated by choosing another pc* and obtaining a new model until the corresponding chemical composition is as close as possible to the one obtained spectroscopically. The obtained results are presented in Table 1.

In this Table the pressure (P) is expressed in units of 10^{18} dyne/cm^2, the temperature (T) in units of 10^6 K, the density ρ in gr/cm^3, q is the reduced mass and f is the reduced luminosity. After the connection of the solution of the radiative nucleus to the one of the convective zone the following values are obtained:

Table 1.

x	P	q	f	T	ρ
0.000	0.1702	0.000	0.000	14.9061	90.8445
0.0058	0.1700	0.1294E-4	0.127E-3	14.9003	90.777
0.0232	0.1671	0.8226E-3	0.789E-2	14.8128	89.777
0.0465	0.1583	0.6442E-2	0.067	14.5389	86.628
0.0814	0.1364	0.0325	0.2423	13.8304	78.492
0.1046	0.1183	0.0654	0.4132	13.2013	71.2911
0.1512	0.0806	0.1693	0.72912	11.7211	54.6589
0.2035	0.0454	0.3293	0.9205	9.9663	36.3029
0.2500	0.0248	0.4807	0.9788	8.5142	23.2447
0.3024	0.1175E-1	0.6332	0.9959	7.0910	13.1845
0.3547	0.5317E-2	0.7530	0.9993	5.8997	7.1688
0.4013	0.2580E-2	0.8316	0.9998	5.0150	4.0926
0.4536	0.1133E-2	0.8937	0.9999	4.1845	2.1551
0.5001	0.5434E-3	0.9311	0.9999	3.5667	1.2118
0.5525	0.2361E-3	0.9589	0.9999	2.9808	0.6301
0.6045	0.1014E-3	0.9763	0.9999	2.4879	0.3244
0.6513	0.4712E-4	0.9859	0.9999	2.1124	0.1774
0.7037	0.1931E-4	0.9925	0.9999	1.7473	0.0878
0.7502	0.6412E-5	0.9960	0.9999	1.4648	0.0456
0.8026	0.3104E-5	0.9981	0.9999	1.1871	0.0207
0.8607	0.9199E-5	0.9993	0.9999	0.9280	0.0078
0.8956	0.4158E-6	0.9996	0.9999	0.8110	0.0040

$$p_c^* = 0.68051331817$$
$$E = 0.92$$
$$X = 0.628,$$
$$Y = 0.325,$$
$$Z = 0.047$$

Schwarzschild obtained for the solar model having the chemical composition of:

$$X = 0.60,\ Y = 0.344,\ Z = 0.056$$

the following values:

$$E = 1.02,\ x_i = 0.887,\ q_i = 0.9997,\ T_i = 0.8 \cdot 10^6 \text{ K},$$
$$T_c = 15 \cdot 10^6 \text{ K},\ \rho_c = 87 \text{ g/cm}^3$$

and in the present paper I obtained:

$$X = 0.628,\ Y = 0.325,\ Z = 0.047$$

the following values:

$$E = 0.92,\ x_i = 0.8956,\ q_i = 0.9996,\ T_i = 0.811 \cdot 10^6 \text{ K},$$
$$T_c = 14.9061 \cdot 10^6 \text{ K},\ \rho_c = 90.8445 \text{ g/cm}^3.$$

The conclusion is that the results are similar even if the way of solving is totally different.

At solving the system (13), using the boundary conditions in the centre of the star (20) there appear indeterminacy in the form of 0/0. I have proposed the Taylor's series $\sum a_n x^n$ for the integration of this system:

$$\begin{aligned} p(x) &= p_0 + A_1 x + A_2 x^2 + A_3 x^3 + ... \\ q(x) &= B_1 x + B_2 x^2 + B_3 x^3 + ... \\ f(x) &= C_1 x + C_2 x^2 + C_3 x^3 + ... \\ t(x) &= 1 + D_1 x + D_2 x^2 + D_3 x^3 + ... \end{aligned} \qquad (28)$$

where $p_0 = p_c^*$, and I take x instead of x^* for an easier use. I have assumed that the pressure $p(x)$, the temperature $t(x)$, the luminosity $f(x)$ and the mass $q(x)$ are continuous functions and using the series (28) in (13) I have obtained their expressions given by (21).

Then I have showed the classical methods of numeric integration used to solve such a system (the formulae 22-24) using the successive approximations up to $\left| {V_{k+1}}^{(n+1)} - {V_{k+1}}^{(n)} \right| < 10^{-11}$.

Schwarzschild used the logarithmic variables for the system (13) transforming the indeterminacy in the form of 0/0 in other indeterminacies in the form of ∞/∞, and Sears used the mass $m{=}M(r)/M$ as independent variable.

When the boundary conditions at the surface of the star are used:

$$\begin{aligned} x &= 1, \\ f &= 1, \\ q &= 1, \\ t &= 0, \\ p &= 0 \end{aligned} \qquad (29)$$

the system (9), which corresponds to the convective cover, has also the indeterminacy in the form of 0/0. Using the series of powers, I have obtained for the convective cover the expressions (25) and (26) and I have shown how the formulae (27) are used to continue the integration of the system (9). In conclusion this way of mathematical and numerical approaching permits obtaining any homogeneous stellar model, which has a radiative nucleus and a convective cover. The papers quoted in the text were consulted at the writing of this paper. Other papers quoted in the References are recommended to be read for a better understanding of the studied theme.

The values of the constants which appear in the paper are:

$$G = 6.672 \cdot 10^{-8} \text{ cm}^3\text{g}^{-1}\text{s}^{-1};$$
$$R = 6.96 \cdot 10^{10} \text{ cm};$$
$$H = 1.6725 \cdot 10^{-24} \text{ g};$$
$$M = 1.9891 \cdot 10^{33} \text{ g};$$
$$k = 1.3805 \cdot 10^{-16} \text{ erg/K};$$
$$\mu = 4/(3 + 5X - Z);$$
$$a = 7.564 \text{ erg}\cdot\text{cm}^{-2}\text{deg}^{-4};$$
$$c = 2.99792458 \text{ cm}\cdot\text{s}^{-1};$$
$$L = 3.12 \cdot 10^{33} \text{ erg}\cdot\text{s}^{-1}.$$

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ХОМОГЕНИ МОДЕЛ ЗВЕЗДЕ ЧИЈИ ЈЕ ХЕМИЈСКИ САСТАВ X = 0.628 И Z = 0.047

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Приказан је модел са масом једнаком Сунчевој маси, са уделом водоника X = 0.628, уделом хелијума Y = 0.325 и уделом метала Z = 0.047. Овај модел одговара младим звездама које припадају Популацији 1.