ON THE GEODESICS FOR A SPHERICALLY SYMMETRIC DILATON BLACK HOLE

C. Blaga\textsuperscript{1} and P. A. Blaga\textsuperscript{2}

\textsuperscript{1} Astronomical Observatory, Cireșilor 19, 3400 Cluj-Napoca, Romania
\textsuperscript{2} "Babeș-Bolyai" University, Faculty of Mathematics and Computer Sciences, 1, Kogălniceanu Street, 3400 Cluj-Napoca, Romania

(Received: May 7, 1998)

SUMMARY: In this paper we shall investigate the timelike geodesics for an extremal, spherically symmetric, massless dilaton black hole, using an exact solution obtained by Gary Horowitz.

1. INTRODUCTION

In classical general relativity, the geometry of a static, charged, spherically symmetric black hole is described by the well-known Reissner-Nordström solution. However, if string theory is to be used for the description of nature, then, in the low energy limit of this theory, the action includes, besides the pure gravitational part, a minimally coupled scalar field, the dilaton. Horowitz (1993) showed that, by applying to the Schwarzschild solution a Harrison-like transformation, we can obtain a metric which is a solution of the Einstein-Maxwell-dilaton field equations and a not very difficult investigation reveals the fact that it is a static, spherically symmetric solution, corresponding to a charged, massless dilaton. The line element for this solution is given by

\begin{equation}
\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + \\
&+ r\left(r - \frac{Q^2}{M}\right)[d\theta^2 + \sin^2 \theta d\varphi^2]
\end{aligned}
\end{equation}

In a previous paper (Blaga, Blaga, 1996), we were able to show that the geodesics equation for this metric is separable and we obtained the following generating function

\begin{equation}
S = -\frac{1}{2}\delta_1 - Et + L_z + \int^r \sqrt{\Delta} \, dr + \int^\theta \sqrt{\Theta} \, d\theta,
\end{equation}

in curvature coordinates. Here

\begin{equation}
\delta_1 = \begin{cases} 
-1 & \text{for timelike geodesics} \\
0 & \text{for null geodesics} \\
1 & \text{for spacelike geodesics}
\end{cases}
\end{equation}

\begin{equation}
\Delta = r\left(r - \frac{Q^2}{M}\right)\left(1 - \frac{2M}{r}\right)
\end{equation}

\begin{equation}
\Theta = Q - \frac{L_z^2}{\sin^2 \theta}
\end{equation}

55
\( Q \) is the fourth constant of the motion (besides \( \delta_1, E \)
and \( L_z \)), it is a separation constant and it was first
introduced by Brandon Carter, when he studied the
motion around a Kerr-Newman black hole.

2. THE GEODESIC EQUATIONS

Starting from the complete integral (2), by
using the Hamilton-Jacobi theorem, we recast
the geodesics equations in the following first order form:

\[
d\lambda = \frac{r \left( r - \frac{Q^2}{r^2} \right) dr}{\sqrt{R\Delta}} \tag{4a}
\]

\[
dt = \frac{r \left( r - \frac{Q^2}{r^2} \right) E}{(1 - \frac{2M}{r}) \sqrt{R\Delta}} dr \tag{4b}
\]

\[
d\varphi = \frac{L_z}{\sin^2 \theta \sqrt{\Theta}} d\theta \tag{4c}
\]

\[
dr = \frac{\sqrt{\Delta \sqrt{r}}}{\sqrt{\Theta}} \tag{4d}
\]

allowing us the study of the motion in this field.

The quantity \( Q \) is related to the electrical
charge of the black hole. Horowitz emphasized that
we are actually dealing with a black hole (and not
a naked singularity) only for \( Q^2 \leq 2M^2 \). We shall
consider hereafter only the case \( Q^2 = 2M^2 \), referred
to as the extremal case.

The timelike geodesics from the equatorial
plane \( \theta = \pi/2 \) are described by the equations (4c)
and (4d).†

Passing from the variable \( r \) to the variable \( u = \frac{1}{r} \) we get:

\[
\left( \frac{du}{d\varphi} \right)^2 = (2Mu - 1)^2 \left( -u + \frac{2M}{L_z^2}u + \frac{E^2}{L_z^2} - 1 \right) \equiv f(u), \tag{5}
\]

or, passing back to \( r \) and performing the square root,

\[
d\varphi = \pm \frac{L_z dr}{\sqrt{|r - 2M| \sqrt{r^2(2E^2 - 1) + 2Mr - L_z^2}}} \tag{6}
\]

It is now clear how to use the equation (6) to
determine the behaviour of geodesics. For each set of
values of the parameters \( (M, E, L_z) \) the only motions
that are allowed are those for which the argument of
the square root is (strictly) positive.

What is interesting to note is that in the case
of the dilaton black hole, no geodesic actually is
passing through the event horizon \( r = 2M \), which
means that no uncharged test particle can reach the
singularity in a free fall. This is not a contradiction
with the fact that we are dealing with a black
hole, because the singularity can be reached in a non-
geodesic motion.

We were able to integrate the equation (6) and
we obtained the following results, for different values
of the quantities:

\[
a = 4M^2E^2 - L^2 \text{ and } \Delta = M^2 + L^2(E^2 - 1).
\]

Using also the notations

\[
I = \pm (\varphi - \varphi_0) \frac{2a}{L} \sqrt{2a} \sin(\sqrt{a} r), \tag{6.1}
\]

\[
b = 2M(2E^2 - 1), \text{ and } c = E^2 - 1,
\]

we have the following cases:

(i) \( a < 0 \) and \( \Delta > 0 \) the solution has the form

\[
r = 2M - \frac{2a}{b} \frac{2a}{b} \sqrt{2a} \sin(\sqrt{a} r), \tag{6.2}
\]

(ii) \( a = 0 \), if \( bx + c > 0 \) the solution reads:

\[
r = 2M + \frac{4b}{b^2I^2 - 4c}, \tag{6.3}
\]

(iii) \( a > 0 \) and \( \Delta \) arbitrary. The solution is

\[
r = 2M + \frac{4\varepsilon \alpha \varepsilon \sqrt{4a}}{(\alpha \varepsilon \alpha \varepsilon - 4a)^2 - 4ac}
\]

where

\[
\varepsilon = \text{sgn}(2ax + b + 2\sqrt{a(ax^2 + bx + c)})
\]

It goes without saying, but has to be said that
only selected parts of the curves described by the
equations (6.1)–(6.3) are actually geodesics.

The general characteristics of the geodesics
will be dealt with in a forthcoming paper. What
we want here is simply to illustrate the shape of them
for some set of parameters.

3. EXAMPLES

We selected for this paper a number of four
sets of values for the parameters \( (M, E, L_z) \), to illustrate
the shape of the timelike geodesics (see Figures
1–4).

† It is not difficult to see that a geodesics for
which at a given moment \( t_0 \), \( \theta = \frac{\pi}{2} \) \( \text{ and } \dot{\theta} = 0 \), does
not leave the equatorial plane.
Fig. 1. $M = 1, E = \sqrt{2}, L_z = 4$ (case (i)).

Fig. 2. $M = 0.9, E = 0.5, L_z = 1.5$ (case (i)).

Fig. 3. $M = 1, E = 0.75, L_z = 1.5$ (case (ii)).

Fig. 4. $M = 1, E = 0.5, L_z = 1$ (case (ii)).
Fig. 5. $M = 1, E = \frac{1}{\sqrt{2}}, L_z = 1$ (case (iii)).

Fig. 6. $M = 1, E = 2, L_z = 2$ (case (iii)).

Fig. 7. $M = 1, E = 2, L_z = 4$ (case (ii)).

REFERENCES


О ГЕОДЕЗИЦИМА ЗА СФЕРНО СИМЕТРИЧНУ БЕЗМАСЕНУ ЦРНУ ЈАМУ

С. Блаза¹ и Р. А. Блаза²

¹ Astronomical Observatory, Cireșilor 19, 3400 Cluj-Napoca, Romania
² "Babeș-Bolyai" University, Faculty of Mathematics and Computer Sciences, 1, Kogălniceanu Street, 3400 Cluj-Napoca, Romania

УДК 524.882
Претходно саопштеве

У овом раду испитујемо временске геодезијске линије у екстремној, сферно симетричној, безмасеној црној јами, користећи једно тачно решење које је добио Gary Horowitz.