THE ZONAL SATELLITE PROBLEM. I.
NEAR-COLLISION FLOW

V. Mioc and M. Stavinschi

Astronomical Institute of the Romanian Academy
Str. Cuțitul de Argint 5, RO-75212 Bucharest 28, Romania

(Received: July 15, 1998)

SUMMARY: The force field described by a potential function of the form
\[ U = \sum_{k=1}^{n} a_k/r^k \] (r = distance between particles, \( a_k \) = real parameters) models various
concrete situations belonging to astronomy, physics, mechanics, astrodynamics, etc.
The two-body problem is being tackled in such a field. The motion equations and the
first integrals of energy and angular momentum are established. The McGehee-type
coordinates are used to blow up the collision singularity and to paste the resulting
manifold on the phase space. The flow on the collision manifold is depicted. Then,
using the rotational symmetry of the problem and the angular momentum integral,
the local flow near collision is described and interpreted in terms of physical motion.

1. INTRODUCTION

This paper starts a research intended to bring
a unifying standpoint for several problems of particle
nonlinear dynamics. Many classical models, as those
of Kepler, Manev, Schwarzschild, Fock, Coulomb,
Van der Waals, etc., are included. We deal with
two-body problems associated to potentials of the
type \( U = \sum_{k=1}^{n} a_k/r^k \), in which \( r \) is the distance
between particles, whereas \( a_k, k = 1, n \), are real con-
stants. Since the best known potential of this kind
is that represented by the zonal part of a planetary
gravitational potential, we shall assign the name of
zonal satellite problem to this class of problems (see
also Cid et al. 1983).

The study of the zonal satellite problem is
important especially for physics, mechanics, astron-
omy, astrodynamics, and not only. Its importance
is emphasized by the multitude of concrete situa-
tions modellable in this way. It is clear that \( a_1 > 0, \)
\( a_k = 0, (k = 2, n) \), features the Newtonian field (for
a photogravitational field with Newtonian gravita-
tional component, \( a_1 \) can equally be negative). The
motion in Manev’s field \( (a_1 > 0, a_2 > 0, a_3 = 0, k = 3, n) \) or in Manev-type fields \( (a_1, a_2 \in R, a_3 = 0, k = 3, n) \) also belongs to this model (see, e.g., Di-
acu 1993, 1996; Diacu et al. 1995, Mioc and Stoica
1995a,b, 1996, 1997; Delgado et al. 1996; Stoica and
Mioc 1997b). The Schwarzschild \( (a_1 > 0, a_2 = 0, \)
\( a_3 > 0, a_k = 0, k = 4, n) \) or Schwarzschild-type
\( (a_1, a_3 \in R, a_2 = 0, a_k = 0, k = 4, n) \) models consti-
tute further examples (see, e.g., Moeckel 1992; Stoica
and Mioc 1997a). Moreover, the zonal satellite prob-
lem models the motion in other relativistic fields, as
for instance Fock’s one (Mioc 1994), or that featured
by the Reissner-Nordström metric. The Coulombian
field (Sommerfeld 1951; Belenkii 1981) also joins the
model. Implications in astrophysics (e.g. Stoica
and Mioc 1997a), mechanics (e.g. Moser 1975; McGehee
1981; Diacu et al.), celestial mechanics and dynamical
astronomy (Blaga and Mioc 1992; Diacu et al.
1995; Mioc and Stoica 1995c; Stoica and Mioc 1996, 1997b), even in atomic physics (Diacu 1993), are to be added to the above arguments.

The present paper attempts to provide a first qualitative insight into the model of the zonal satellite problem. In Section 2 we reduce the problem to a central force problem, write the equation of motion, and point out the first integrals of energy and angular momentum. The qualitative analysis starts with Section 3, in which the powerful tool of McGehee’s (1974) technique is used to blow up the collision singularity and to provide new, regularized equations of motion.

In Section 4, consequent on McGehee’s transformations, the collision manifold \((M_0)\) is passed, instead of the singularity, on the phase space. It proves to be a 2D torus imbedded in the full 4D phase space if \(a_n > 0\), and the empty set (collisionless motion) for \(a_n < 0\). The flow on the \(M_0\) torus (consisting of two circles of degenerate equilibria, connected by heteroclinic orbits moving from the upper circle to the lower one) is deprived of physical significance. However, due to the continuity of solutions with respect to initial data, its study provides valuable informations about the local flow near collision.

The behaviour of the solutions in the immediate neighbourhood of collision is tackled in Section 5. Exploiting successively the rotational symmetry of the global flow (characteristic to the zonal satellite problem) and the angular momentum integral, we restrict the full phase space to the 3D reduced phase space, then to dimension 2 (the phase plane - PP). This allows a clear description of the local structure of PP near collision, for the whole possible interplay between the parameters \(a_{n-1}\) and \(a_n\). The description is completed by a physical interpretation of the respective phase curves.

2. EQUATIONS OF MOTION AND FIRST INTEGRALS

It is clear that the potential we deal with is central, therefore the associated two-body problem can be reduced to a central force problem. The motion is confined to a plane, so we fix one particle (hereafter centre) at the origin of this plane \(R^2\), and study the relative motion of the other particle. Within this framework, denoting by \(q = (q_1, q_2) \in R^2\) the position (or configuration) vector of the particle, the potential reads

\[
U(q) = \sum_{k=1}^{n} a_k / |q|^k. \tag{1}
\]

Let us introduce the momentum vector \(p = (p_1, p_2) \in R^2\), and denote by \(T(p) = |p|^2 / 2\) the kinetic energy of the particle. In canonical formalism, the equations of motion read

\[
\dot{q} = \partial H(q, p) / \partial p, \tag{2}
\]

\[
\dot{p} = - \partial H(q, p) / \partial q.
\]

defining a vector field on the phase space \(Q \times P\), where \(Q = R^2 \setminus \{(0, 0)\}\) is the configuration space, whereas \(P = R^2\) is the momentum space. The Hamiltonian function has the form

\[
H(q, p) = T(p) - U(q) = |p|^2 / 2 - \sum_{k=1}^{n} a_k / |q|^k. \tag{3}
\]

Standard results of the differential equations theory ensure, for given initial conditions \((q, p)(0) \in Q \times P\), the existence and uniqueness of a real analytic solution \((q, p)\) of equations (2), defined locally on some time interval \((t^-, t^+)\), \(-\infty \leq t^- < 0 < t^+ \leq \infty\). Due to the symmetry, we may study, without loss of generality, the properties of the solution on \([0, t^+]\) only (i.e., the motion in the future). It is clear that \([0, t^+]\) can now be extended to a maximal interval \([0, t^*]\), \(t^+ \leq t^* \leq \infty\).

Using the standard technique, we find that the Hamiltonian has the property

\[
H(q, p) = h / 2 = \text{const}, \tag{4}
\]

namely is a first integral of the system (called the integral of energy; \(h\) is the energy constant). The field being central, the angular momentum is conserved, hence we can obtain another first integral

\[
L(q, p) = \vec{q}_1 p_2 - q_2 p_1 = C = \text{const}, \tag{5}
\]

where \(C\) is the constant of angular momentum.

In an explicit form, the equations of motion are

\[
\dot{q} = p, \tag{6}
\]

\[
\dot{p} = - \sum_{k=1}^{n} (ka_k / |q|^{k+2})q, \tag{7}
\]

with the energy relation

\[
|p|^2 - 2 \sum_{k=1}^{n} a_k / |q|^k = h. \tag{8}
\]

3. McGehee’s Transformations

The potential (1) has an isolated singularity for \(t = t^* < \infty\), at \(q = (0, 0)\). Using a Painlevé-type criterion (e.g. Diacu 1992), it is easy to prove that this singularity corresponds to the collision particle-centre. To remove it and to regularize equations (6), we shall resort to McGehee-type transformations of the second kind (McGehee 1974).

For the first step we use the real analytic diffeomorphism

\[
Q \times P \times [0, t^*) \to (0, \infty) \times [0, 2\pi] \times R \times R \times [0, t^*) \times (q_1, q_2), (p_1, p_2); t) \to (r, \theta, \xi, \eta; t)
\]
defined by
\[ r = |\mathbf{q}|, \]
\[ \theta = \arctan(q_2/q_1), \]
\[ \xi = \hat{r} = (q_1p_1 + q_2p_2)/|\mathbf{q}|, \]
\[ \eta = r\hat{\theta} = (q_1p_2 - q_2p_1)/|\mathbf{q}|, \]
which introduces the standard coordinates and the polar components of the velocity. In variables (8), the motion equations acquire the form
\[ \dot{r} = \xi, \]
\[ \dot{\theta} = \eta/r, \]
\[ \dot{\xi} = \eta^2/r - \sum_{k=1}^{n} ka_k/r^{k+1}, \]
\[ \dot{\eta} = -\xi\eta/r, \]
which while the first integrals (7) and (5) become respectively
\[ \xi^2 + \eta^2 - 2\sum_{k=1}^{n} a_k/r^k = h; \] (10)
\[ r\xi = C. \] (11)
The singularity corresponds now to \( r = 0 \). For the second step, we have to scale down the components of the velocity via the real analytic diffeomorphism
\[ (0, \infty) \times [0, 2\pi] \times \mathbb{R} \times \mathbb{R} \times [0, t^*) \rightarrow (0, \infty) \times [0, 2\pi] \times \mathbb{R} \times \mathbb{R} \times [0, t^*) \]
\[ (r, \theta, \xi, \eta; t) \rightarrow (r, \theta, x, y; t) \]
defined by
\[ x = r^{n/2}\xi, \]
\[ y = r^{n/2}\eta. \] (12)
The equations of motion (9) now read
\[ \dot{x} = x/r^{n/2}, \]
\[ \dot{y} = y/r^{n/2+1}, \]
\[ \dot{\xi} = (nx^2/2 + y^2)/r^{n/2+1} - \sum_{k=1}^{n} ka_k/r^{k+1-n/2}, \]
\[ \dot{\eta} = (n/2 - 1)xy/r^{n/2+1}, \]
whereas the first integrals (10) and (11) become respectively
\[ x^2 + y^2 = hr^n + 2\sum_{k=1}^{n} a_k r^{n-k}; \] (14)
\[ y^2 = C^2 r^{n-2}. \] (15)
The singularity at \( r = 0 \) still persists. To remove it, we introduce the timelike variable \( s \) through the real analytic diffeomorphism
\[ (0, \infty) \times [0, 2\pi] \times \mathbb{R} \times \mathbb{R} \times [0, t^*) \rightarrow (0, \infty) \times [0, 2\pi] \times \mathbb{R} \times \mathbb{R} \times [0, t^*) \]
\[ (r, \theta, x, y; t) \rightarrow (r, \theta, x, y; s) \]
defined by
\[ ds = r^{-n/2-1}dt. \] (16)
With (16), and keeping by abuse the same notations for the new functions of the fictitious time \( s \), the motion equations (13) become
\[ \dot{r'} = rx, \]
\[ \dot{\theta'} = y, \]
\[ \dot{x'} = nx^2/2 + y^2 - \sum_{k=1}^{n} ka_k r^{n-k}, \]
\[ \dot{y'} = (n/2 - 1)xy, \]
where \( r' = d/ds \). Obviously, the integrals (14) and (15) keep their expressions.

4. COLLISION MANIFOLD
Both the equations of motion (17) and the first integrals (14)-(15) are now well defined for the boundary \( r = 0 \). This means that the phase space can be analytically extended to contain the manifold
\[ \{ (r, \theta, x, y) | r = 0, \} \]
which is invariant under the flow because \( r' = 0 \) for \( r = 0 \). The relations (14) and (15) also extend smoothly to this boundary.
Having in view (14), let us define the constant energy manifold
\[ \{ (r, \theta, x, y) | x^2 + y^2 = hr^n + 2\sum_{k=1}^{n} a_k r^{n-k}, \} \]
which corresponds to a fixed level of energy. Now we are able to define the collision manifold
\[ M_{0} = M_{\text{col}} \cap M_{h} = \{ (r, \theta, x, y) | r = 0, \theta \in S^1, x^2 + y^2 = 2a_n \}. \] (18)
Since the last term in the expansion of the potential (1) must obviously be nonzero, the case \( a_n = 0 \) will not be considered. For \( a_n < 0 \), \( M_0 \) is the empty set; the particle cannot encounter collisions. Analogous results were established by Saari (1974), or Stoica and Mioic (1997a) within different contexts.
Consider hence \( a_n > 0 \); by (18), \( M_0 \) is a 2D cylinder in the 3D space of the coordinates \((\theta, x, y) \in S^1 \times \mathbb{R} \times \mathbb{R}\) (Figure 1). But, since \( \theta \in S^1 \) (the segment \([0, 2\pi]\) with the ends points pasted together), the \( M_0 \) cylinder may be identified with a 2D torus, both actually being imbedded in the 4D full phase space of the McGehee- type coordinates \((r, \theta, x, y)\).
So, by means of McGehee’s technique, we have blown up the singularity and pasted the \( M_0 \) torus, instead of it, on the phase space. Having in view (16), \( M_0 \) is formed only by equilibria of the flow on the full phase space. In other words, every collisional phase trajectory needs an infinite amount of fictitious time \( s \) to reach \( M_0 \).

In the sequel we shall describe the flow on \( M_0 \). This flow, although deprived of physical significance, provides valuable informations about the behaviour of near-collision orbits (due to the continuity of solutions with respect to the initial conditions).

Using (17) and (14), the vector field on \( M_0 \) reads

\[
\begin{align*}
\theta' &= y, \\
x' &= -(n/2 - 1)y^2, \\
y' &= (n/2 - 1)xy.
\end{align*}
\]

One sees immediately that equations (19) admit the fixed points \((\theta, x, y) = (\theta_c, \pm \sqrt{2a_n}, 0)\), with arbitrary \( \theta \in S^1 \). Explicitly, there are two circles of degenerate equilibria on the \( M_0 \) torus: the upper circle (UC): \((\theta \in S^1, x = \sqrt{2a_n}, y = 0)\), and the lower circle (LC): \((\theta \in S^1, x = -\sqrt{2a_n}, y = 0)\).

By (19), removing the cases \( n = 1 \) (Newton-type field) and \( n = 2 \) (Mane\-type field), one sees that \( x' < 0 \) for \( y 
eq 0 \), which means that all orbits of the flow on \( M_0 \) are heteroclinic and move from UC to LC. To determine the shape of these trajectories, let us introduce the variable \( \alpha \) via \( x = \sqrt{2a_n} \cos \alpha \), \( y = \sqrt{2a_n} \sin \alpha \). Taking into account (19), this leads to \( d\alpha/d\theta = n/2 - 1 \).

To exemplify, the phase curves on \( M_0 \) for \( n = 4 \) are illustrated in Figure 1.

This provides a complete qualitative image of the flow on \( M_0 \).

To add some issues within this framework, focus on formula (15). On the one hand, it is clear that all collisional trajectories eject from UC or tend to LC. On the other hand, collisions occur not only for \( C = 0 \), hence for radial motion (as in the Newtonian case), but also for \( C \neq 0 \). This is the so-called black hole effect (the particle spirals infinitely many times around the centre before collision/after ejection) pointed out by Wintner (1941), McGehee (1981), Diaçu et al. (1995), or Stoica and Mioc (1997a).

As a final remark, in case \( M_0 \) is nonempty, it does not depend on \( h \), hence every total energy level shares this boundary.

5. NEAR-COLLISION FLOW

In this section we shall study the local flow in the neighbourhood of the collision manifold. First observe that \( \theta \) does not appear explicitly in either the regularized vector field (17) or first integrals (14)-(15). We can hence reduce the dimension of the phase space from 4 to 3, by factorizing the flow to \( S^1 \). Of course, every phase curve in this 3D reduced phase space (hereafter RPS) actually is a manifold consisting of the product between the respective orbit and \( S^1 \). The \( M_0 \) torus in full phase space reduces in RPS to the circle \( M_0 = \{ r = 0, x^2 + y^2 = 2a_n \} \).

We can further reduce RPS to dimension 2 by resorting to the integral of angular momentum (15), which relates \( y \) to \( r \). So we obtain the phase plane of the coordinates \((r, x)\) (hereafter PP). It is obvious that what we said above about the orbits in RPS is to be applied to the solutions in PP, too. The \( M_0 \) circle in RPS reduces in PP to the points \( M \) \((r = 0, x = \sqrt{2a_n})\) and \( N \) \((r = 0, x = -\sqrt{2a_n})\).

Under these conditions, the energy integral in PP reads

\[
x^2 = \sum_{j=0}^{n} A_j r^{n-j},
\]

where \( A_0 = h, A_1 = 2a_1, A_2 = 2a_2 - C^2, A_j = 2a_j \) \((j = 3, n)\). It is obvious that we consider \( a_n \neq 0 \) and very small \( r \) (neglecting the terms in \( r^i \), \( i \geq 2 \)).

The local phase portrait for \( a_n > 0 \) is given in Figure 2a. For \( a_{n-1} < 0 \) we have the curves 1: heteroclinic trajectories which eject from collision, reach a maximum distance \( r_{\text{max}} = -a_n/a_{n-1} \), then tend back to collision. For \( a_{n-1} = 0 \), we have the curves 2 and 2’ (which eject from collision or tend to collision, respectively). For \( a_{n-1} > 0 \), the phase portrait consists of the curves 3 and 3’, which have similar significances as the curves 2 and 2’, respectively. Of course, the ejection/collision is asymptotic. The phase portrait also contains the points \( M \) and \( N \): degenerate orbits which remain forever in collision (in this first approximation with local character).

In case \( a_n < 0 \), the motion is collision-free. If \( a_{n-1} < 0 \) the real motion is impossible in our approximation. If \( a_{n-1} > 0 \), the phase portrait in PP is given in Figure 2b: the phase trajectories approach the centre, reach a minimum distance \( r_{\text{min}} = -a_n/a_{n-1} \), then move away from the centre.

The near-collision physical orbits behave like the above described PP trajectories. It is easy to see that the motion is radial for \( C = 0 \), and spiral for \( C \neq 0 \) (with black hole effect at collision/ejection in the case of collisional motion).

These results provide a qualitative picture of the phase and physical motion in the neighbourhood of collision within the framework of the zonal satellite problem.
Fig. 2. The near-collision flow in PP for: (a) $a_n > 0$ (b) $a_n < 0$.

REFERENCES

ПРОБЛЕМ ЗОНСКИХ САТЕЛИТА. I.
ПРОТОК У ОБЛАСТИ БЛИЗУ СУДАРА

В. Миок и М. Ставински

Astronomical Institute of the Romanian Academy
Str. Cuțitul de Argint 5, RO-75212 Bucharest 28, Romania

УДК 521.17
Оригинална научна рад

Поле сила описано потенцијалом облика

\[ U = \sum_{k=1}^{n} \frac{a_k}{r^k} \]  

(\( r \) - растојање између материјалних тачака, \( a_k \) = реални параметри) моделује разне конкретне ситуације које спадају у астрономију, физику, механику, астродинамику итд. У таквом поле се разматра проблем двају тела. Установљени су једначине кретања и први интеграли енергије и момента импулса. Коришћене су координате МекГесовог типа у циљу одстрановања сударне сингуларности и пресликавања резултујућег скупа на фазни простор. Описано је протицање на сударном скупу. Тада, коришћењем обртне симетрије проблема и интеграла момента импулса описује се и интерпретира локални проток у области близу судара преко физичког кретања.